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A two-dimensional analogue of Padé approximant theory

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Abstract. We introduce a formulation for two-dimensional Padé approximant theory. We prove a theorem on convergence under the stringent assumption of uniform boundedness. Finally we discuss some general problems connected with rational functions in several variable theory.

1. Introduction

In this paper we attempt to formulate the basis for a two-dimensional rational approximant which contains the Padé solution in one dimension.

Rational approximants to a holomorphic function in any dimensions are extremely useful for the following reasons:

(i) They always contain a subsequence which converges with considerable acceleration to the holomorphic function. This has been observed numerically.

(ii) They, in general, have a much larger domain of convergence than the homomorphic function they approximate and hence they can be continued analytically outside the domain of convergence of the holomorphic function, except where the latter has a natural boundary.

Given a function f defined in the complex two-dimensional euclidean space $\mathbb{C}^2$, which is analytic at a point $\zeta_0 = (z_{10}, z_{20}) \in \mathbb{C}^2$, we have within some neighbourhood $U_{\zeta 0}$

$$f(\zeta) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} c_{\mu\nu}(z_1 - z_{10})^{\mu}(z_2 - z_{20})^{\nu} \tag{1.1}$$

where $\zeta = (z_1, z_2) \in U_{\zeta_0}$ and $c_{\mu\nu}$ are all known. In principle we can construct a suitable sequence of rational approximants to f in U_{ζ_0}, knowing the c, which, according to (i) and (ii), has a subsequence convergent to f in a domain larger than U_{ζ_0}.

However, rational functions being meromorphic functions raise some of the problems associated with such functions in the two or more complex variables. We shall defer discussion of some of these problems until the conclusion. For simplicity we shall tackle cases where $\zeta_0 = (0, 0)$.

The pattern of the work is as follows: in § 2 we present the definition and its consequences; in § 3 we present a simple convergence theorem under the stringent assumption of uniform boundedness; in § 4, we discuss some general problems to be borne in mind when dealing with approximants in several complex variables. Appendix 1 contains some examples of equations for solving the coefficients of the approximant, while appendix 2 contains a table of a few terms in the 'tensor' table for a known function, and also a geometric representation for the 'tensor' table.

2. Definition and consequences

Let $f(\zeta)$ be analytic in some domain $\mathscr{D} \subset \mathbb{C}^2$ and be holomorphic in some neighbourhood U of the origin, with $U \subset \mathscr{D}$, then $f(\zeta)$ has a Taylor expansion

$$f(\zeta) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} c_{\mu\nu} z_1^{\mu} z_2^{\nu} \tag{2.1}$$

where $c_{\mu\nu} \in \phi$ (space of complex numbers) and $c_{00} \neq 0$. The polynomials $P_{NN'}(\zeta)$ and $Q_{MM'}(\zeta)$ which are chosen relatively prime† in U, defined by

$$P_{NN'}(\zeta) = \sum_{r=0}^{N} \sum_{s=0}^{N'} a_{rs} z_1^r z_2^s \tag{2.2}$$

$$Q_{MM'}(\zeta) = \sum_{r=0}^{M} \sum_{s=0}^{M'} b_{rs} z_1^r z_2^s \tag{2.3}$$

are both holomorphic in U. The quotient of these two polynomials $P_{NN'}(\zeta)/Q_{MM'}(\zeta)$, which is a rational function, is defined only on $U \setminus \mathscr{A}$ where $\mathscr{A}$ is the set of its indeterminate points†.

To be able to define the approximant we have in mind let us examine the relation

$$f(\zeta) - \frac{P_{NN'}(\zeta)}{Q_{MM'}(\zeta)} = \sum_{\kappa,\lambda} \alpha_{\kappa\lambda} z_1^{\kappa} z_2^{\lambda} \tag{2.4}$$

in U where $\kappa \geqslant 0$, $\lambda \geqslant 0$. If we assume $\alpha_{\kappa\lambda} = 0$ for all $0 \leqslant \kappa \leqslant N+M$ and $0 \leqslant \lambda \leqslant N'+M'$ the relation (2.4) leads to the following sets of equations:

$$\sum_{r=0}^{\min(M,\kappa)} \sum_{s=0}^{\min(M',\lambda)} b_{rs} c_{\kappa-r,\lambda-s} = a_{\kappa\lambda} \qquad 0 \leqslant \kappa \leqslant N; 0 \leqslant \lambda \leqslant N' \tag{2.5}$$

$$\sum_{r=0}^{\min(M,\kappa)} \sum_{s=0}^{\min(M',\lambda)} b_{rs} c_{\kappa-r,\lambda-s} = 0 \qquad 0 \leqslant \kappa \leqslant N; N'+1 \leqslant \lambda \leqslant N'+M' \tag{2.6}$$

$$\sum_{r=0}^{\min(M,\kappa)} \sum_{s=0}^{\min(M',\lambda)} b_{rs} c_{\kappa-r,\lambda-s} = 0 \qquad N+1 \leqslant \kappa \leqslant N+M; 0 \leqslant \lambda \leqslant N' \tag{2.7}$$

$$\sum_{r=0}^{\min(M,\kappa)} \sum_{s=0}^{\min(M',\lambda)} b_{rs} c_{\kappa-r,\lambda-s} = 0 \qquad N+1 \leqslant \kappa \leqslant N+M; N'+1 \leqslant \lambda \leqslant N'+M'. \tag{2.8}$$

Each of the equations (2.5), (2.6), (2.7) and (2.8) represents a set of rectangular block equations which can be joined to fill up a larger rectangle below.

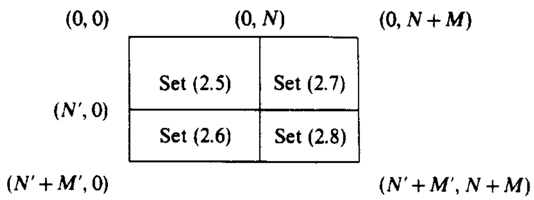

Figure 1.

† See § 4.

The total number of equations from (2.5) to (2.8) is $(N+M+1)\times(N'+M'+1)$. We shall call the linear space of these equations $\mathbb{S}$. The linear spaces spanned by the a and the b respectively shall be denoted $\mathbb{A}$ and $\mathbb{B}$, where $\dim(\mathbb{A}) = (N+1)\times(N'+1)$ and $\dim(\mathbb{B}) = (M+1)\times(M'+1)$. In general, it turns out that $\dim(\mathbb{S}-\mathbb{A}) > \dim \mathbb{B}$, which simply means the number of independent equations in $\mathbb{S}-\mathbb{A}$ exceeds that in $\mathbb{B}$. But since the number of independent variables of the independent equations in $\mathbb{S}-\mathbb{A}$ is equal to $\dim(\mathbb{B})$, all the b in $\mathbb{B}$ must vanish and hence the a must also vanish. In other words the approximant is simple indeterminate at each point. This is a direct consequence of the choice of $\alpha_{\kappa\lambda}$ above. Thus to define a meaningful approximant not all the $\alpha_{\kappa\lambda}(0 \leqslant \kappa \leqslant N+M, 0 \leqslant \lambda \leqslant N'+M')$ must vanish. We modify (2.4) into the form

$$Q_{MM'}(\zeta)f(\zeta)-P_{NN'}(\zeta) = B_{NN'MM'}(\zeta)+O(z_1^{N+M+1})O(z_2^{N'+M'+1}). \tag{2.9}$$

We now introduce the definition of a rational approximant of type B^1 to $f(\zeta)$ holomorphic on U as follows.

Definition. Let $[N,N';M,M']_f(\zeta) = P_{NN'}(\zeta)/Q_{MM'}(\zeta)$. We say $[N,N';M,M']_f(\zeta)$ is a rational approximant type B^1 to $f(\zeta)$ in $U\diagdown\mathscr{A}$ if $B^1_{NN'MM'}(\zeta)$ given by

$$\begin{aligned} B^1_{NN'MM'}(\zeta) = &\sum_{\kappa=1}^{N}\sum_{\lambda=N'+1}^{N'+M'} z_1^\kappa z_2^\lambda\left(\sum_{r=0}^{\min(M,\kappa)}\sum_{s=0}^{\min(M',\lambda)} b_{rs}c_{\kappa-r,\lambda-s}\right)\\ &+\sum_{\kappa=N+1}^{N+M}\sum_{\lambda=1}^{N'} z_1^\kappa z_2^\lambda\left(\sum_{r=0}^{\min(M,\kappa)}\sum_{s=0}^{\min(M',\lambda)} b_{rs}c_{\kappa-r,\lambda-s}\right) \end{aligned} \tag{2.10}$$

is different from zero.

The form of $B_{NN'MM'}(\zeta)$ in (2.9) depends on which $\alpha_{\kappa\lambda}$ do not vanish for $0 \leqslant \kappa \leqslant N+M$ and $0 \leqslant \lambda \leqslant N'+M'$.

In (2.10) we use a specific form of $B_{NN'MM'}(\zeta)$ which we call $B^1_{NN'MM'}(\zeta)$. The effect of this particular choice is to disengage from the equations in $\mathbb{S}$ those unnecessary extra equations leaving us with a set of independent equations,

$$\sum_{r=0}^{\min(M,\kappa)}\sum_{s=0}^{\min(M',\lambda)} b_{rs}c_{\kappa-r,\lambda-s} = a_{\kappa\lambda} \qquad 0 \leqslant \kappa \leqslant N; 0 \leqslant \lambda \leqslant N' \tag{2.11}$$

$$\sum_{s=0}^{\min(M',\lambda)} b_{0s}c_{0,\lambda-s} = 0 \qquad N'+1 \leqslant \lambda \leqslant N'+M' \tag{2.12}$$

$$\sum_{r=0}^{\min(M,\kappa)} b_{r0}c_{\kappa-r,0} = 0 \qquad N+1 \leqslant \kappa \leqslant N+M \tag{2.13}$$

$$\sum_{r=0}^{\min(M,\kappa)}\sum_{s=0}^{\min(M',\lambda)} b_{rs}c_{\kappa-r,\lambda-s} = 0 \qquad N+1 \leqslant \kappa \leqslant N+M; N'+1 \leqslant \lambda \leqslant N'+M'. \tag{2.14}$$

Since we have $(M+1)\times(M'+1)-1$ independent equations involving the b alone, we introduce the normalization $b_{00} = 1$ in order to obtain a unique solution for the remaining b. Once this has been done we may proceed to solve for the a in terms of the b and the c. Hence the rational function will be determined in the sense of (2.10). Examples have been included in the appendix 1. In analogy with the Padé table for the rational sequences, we build up a 'tensor' table for the type B^1 approximants $[N,N';M,M'](\zeta)$.

The table turns out to fill up an infinite cubic figure which we have attempted to represent. A convenient way to describe the location of elements in the infinite configuration is first to locate the plane on which the elements lie. We characterize such planes by the suffices M', N' and denote them by $B_{N'M'}$. We shall, from now on, refer to them as the B-planes. The following is an array of the B-planes.

$$\begin{array}{cccc} B_{00} & B_{01} & B_{02} & \cdots \\ B_{10} & B_{11} & B_{12} & \cdots \\ B_{20} & B_{21} & B_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Each B-plane is spanned by

$$\begin{array}{rcccc} B_{rs} = & [0,r;0,s] & [0,r;1,s] & [0,r;2,s] & \cdots \\ & [1,r;0,s] & [1,r;1,s] & [1,r;2,s] & \cdots \\ & [2,r;0,s] & [2,r;1,s] & [2,r;2,s] & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{array}$$

These two tables completely describe any element's position in the 'tensor' table. A geometrical representation of the B-planes is attached in appendix 2. The question of normality of the 'tensor' table will not be discussed here.

Remarks. In introducing the definition of the type B^1 approximant we chose a specific form of $B_{NN'MM'}(\zeta)$ to remove certain subsets of the equations (2.6) and (2.7). It is clear from these equations that there are at most

$$\binom{(N+1)M'}{M'} \times \binom{(N'+1)M}{M}$$

such choices for the $B_{NN'MM'}(\zeta)$ one can make. However, the form in (2.10) is unique amongst all possible forms $B^{\alpha}_{NN'MM'}(\zeta)$ in that it is the only one which leads to approximants reducible to the Padé approximant on projection from two variables to one. That is,

$$[N,N';M,M']_f(z_1,0) = [N,M]_f(z_1)$$

$$[N,N';M,M']_f(0,z_2) = [N',M']_f(z_2).$$

The definition of the rational approximants type B^1 or B^{α} always includes two variable analytic functions of the form

$$f(z_1,z_2) = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} c_{\mu\nu}\delta_{\mu\nu}z_1^{\mu}z_2^{\nu}$$

for example

(i) $\quad f(z_1,z_2) = \mathrm{e}^{z_1 z_2}$

(ii) $\quad f(z_1,z_2) = \dfrac{1}{1-z_1z_2}$.

We have thus formulated a two-dimensional type B^1 approximant which we show in the next section contains a convergent subsequence in some compact bicylindrical domain if we assume the approximants to be uniformly bounded.

There are, however, other alternatives (John and Lutterodt 1973, Chisholm 1972, Common and Graves-Morris 1972) in formulating rational approximants in two or several variables.

3. Convergence

The problem of convergence of Padé approximants is generally acknowledged to be a difficult one. Under various restrictions, however, quite a number of useful results have been obtained in the one-dimensional case which may not generalize easily to two or more dimensions.

In the sequel we give an account of generalization of a result obtained by Baker (1965, 1970) for the one-dimensional case under the stringent assumption of uniform boundedness for the elements of the Padé table. This is embodied in the following theorem which is proved in some bicylindrical domain.

Theorem (i) Let $\{P_{k_1k_2}(\zeta)\}$ be any sequence of the 'tensor' table defined in

$$\mathscr{D} \subset \mathbb{C}^2((0,0) \in \mathscr{D})$$

where k_1, k_2, respectively characterize (N, M) and (N', M') with $N > M, N' > M'$.

(ii) $N+M \to \infty$ as $k_1 \to \infty$, $N'+M' \to \infty$ as $k_2 \to \infty$.

(iii) $\{P_{k_1k_2}(\zeta)\}$ is uniformly bounded in $\bar{C}_1 \times \bar{C}_2 \subset \mathscr{D}$ where $C_i = \{z_i : |z_i| < R_i\}$ and $\bar{C}_i$ is the closure of C_i, $i = 1, 2$.

Then there exists a subsequence of $\{P_{k_1k_2}(\zeta)\}$ on a compact sub-domain of $C_1 \times C_2$ which converges uniformly to a holomorphic function f.

Proof. From (iii) there exists $\mathscr{M}$ independent of $z_i \in \bar{C}_i$ $(i = 1, 2)$ such that

$$|P_{k_1k_2}(\zeta)| \leqslant \mathscr{M} \qquad \zeta \in \bar{C}_1 \times \bar{C}_2. \tag{3.1}$$

Since $P_{k_1k_2}(\zeta)$ is a rational function in each variable separately, this implies analyticity in each variable separately within $C_1 \times C_2$. Hence by Osgood's lemma, $P_{k_1k_2}(\zeta)$ is analytic in both variables in $C_1 \times C_2$ and we write

$$P_{k_1k_2}(\zeta) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{k_1k_2}(n_1, n_2) z_1^{n_1} z_2^{n_2}. \tag{3.2}$$

We choose a sub-bicylindrical domain of $C_1 \times C_2$ where $P_{k_1k_2}(\zeta)$ converges absolutely and uniformly, defining the sub-bicylindrical domain $C^{r_1} \times C^{r_2}$ by

$$C^{r_i} = \{z_i \in C_i : |z_i| \leqslant r_i < R_i (i = 1, 2)\}.$$

Within $C^{r_1} \times C^{r_2}$ we have,

$$\left| P_{k_1k_2}(\zeta) - \sum_{n_1=0}^{J_1} \sum_{n_2=0}^{J_2} a_{k_1k_2}(n_1, n_2) z_1^{n_1} z_2^{n_2} \right| \leqslant \frac{\mathscr{M}(r_1/R_1)^{J_1+1}(r_2/R_2)^{J_2+1}}{(1-r_1/R_1)(1-r_2/R_2)} \tag{3.3}$$

where we have used the generalized Cauchy's inequality for fixed k_1, k_2

$$|a_{k_1k_2}(n_1, n_2)| \leqslant \frac{\mathcal{M}}{R_1^{n_1}R_2^{n_2}} \qquad \forall n_1, n_2. \tag{3.4}$$

Since $0 \leqslant r_i/R_i < 1$ $(i = 1, 2)$ given $\epsilon > 0$, there exists J_0 such that

$$\left(\frac{r_i}{R_i}\right)^{J_i} < \left[\left(\frac{R_i}{r_i}-1\right)^2 \frac{\epsilon}{3\mathcal{M}}\right]^{1/2} \qquad \forall J_i > J_0 \qquad (i = 1, 2)$$

and hence for all $J_i > J_0$ $(i = 1, 2)$

$$\left| P_{k_1k_2}(\zeta) - \sum_{n_1=0}^{J_1} \sum_{n_2=0}^{J_2} a_{k_1k_2}(n_1, n_2) z_1^{n_1} z_2^{n_2} \right| < \tfrac{1}{3}\epsilon. \tag{3.5}$$

From (ii) we choose K_1 and K_2 such that wherever $j_1, k_1 > K_1$ and $j_2, k_2 > K_2$

$$N(j_1) > J_1 \qquad N'(j_2) > J_2$$
$$N(k_1) > J_1 \qquad N'(k_2) > J_2\dagger$$

so that

$$\left| \sum_{n_1=0}^{J_1} \sum_{n_2=0}^{J_2} (a_{k_1k_2}(n_1, n_2) - a_{j_1j_2}(n_1, n_2)) z_1^{n_1} z_2^{n_2} \right| < \tfrac{1}{3}\epsilon. \tag{3.6}$$

This is possible from the definitions (2.9) and (2.10) together with $N > M$ and $N' > M'$ by making $B^1_{NN'MM'}(\zeta)$ shrink to zero on the boundary $O(z_1^{N+M+1})$. $O(z_2^{N'+M'+1})$ for sufficiently large N and N'.

We now take $K_0 = \max_{1 \leqslant i \leqslant 2}(K_i)$ and consequently for $k_i, j_i > K_0$ $(i = 1, 2)$ and for $\zeta \in C^{r_1} \times C^{r_2}$ we find

$$\begin{aligned} &|P_{k_1k_2}(\zeta) - P_{j_1j_2}(\zeta)| \\ &\quad \leqslant \left| P_{k_1k_2}(\zeta) - \sum_{n_1=0}^{J_1} \sum_{n_2=0}^{J_2} a_{k_1k_2}(n_1, n_2) z_1^{n_1} z_2^{n_2} \right| + \left| \sum_{n_1=0}^{J_1} \sum_{n_2=0}^{J_2} (a_{k_1k_2}(n_1, n_2) \right. \\ &\quad\quad \left. - a_{j_1j_2}(n_1, n_2)) z_1^{n_1} z_2^{n_2} \right| + \left| P_{j_1j_2}(\zeta) - \sum_{n_1=0}^{J_1} \sum_{n_2=0}^{J_2} a_{j_1j_2}(n_1, n_2) z_1^{n_1} z_2^{n_2} \right| \\ &\quad < \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon \end{aligned} \tag{3.7}$$

from (3.5) and (3.6).

Hence by the generalized principle of convergence (Whittaker and Watson 1962, Rankin 1963) and Vitali's theorem (Gunning and Rossi 1965) on $C^{r_1} \times C^{r_2}$ a subsequence of $\{P_{k_1k_2}(\zeta)\}$ exists which converges uniformly to a holomorphic function f on $C^{r_1} \times C^{r_2}$.

Remark. In view of the requirements of the above theorem diagonal approximants $(N = M = N' = M')$ of the B^1-type approximants are excluded. In fact, the diagonal

† I am grateful to Dr J Gunson for pointing out this choice.

approximants fail to converge except, perhaps, in some very small neighbourhood of the origin. This is demonstrated by the table of appendix 2.

4. Discussion

In § 2 we postponed discussion of the problems associated with functions in several complex variables which are of some importance in studying rational approximants to holomorphic functions, using Taylor expansions in more than one dimension. We shall briefly discuss three of these problems which in a sense are related to the problems of extending local properties of functions to global ones.

The first problem is to do with the relatively prime requirement for the two polynomials P and Q in the rational approximant or function $R = P/Q$, at each point of some neighbourhood where f has a Taylor development. It is true in general, in several complex variables for meromorphic (rational) functions that if the relatively prime condition is satisfied at some point ζ_0, then it is satisfied everywhere (Gunning and Rossi 1965) in some neighbourhood U_{ζ_0}. However, in some domain $\mathscr{D}$ where $U_{\zeta_0} \subset \mathscr{D}$, R may well possess some singularities peculiar only to several complex variables, which invalidate the relatively prime condition. In fact R is indeterminate on the subvariety $\mathscr{A} = \{\zeta \in \mathscr{D} : P = 0 \text{ and } Q = 0\}$ of these singularities. For instance, the rational function R such that $P = z_2 - z_1 + \frac{1}{2}z_1^2 - \frac{1}{2}z_2^2$ and $Q = 1 - z_1 - z_2 + \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ satisfies the relatively prime condition in some neighbourhood of the origin, however, it has a point of $\mathscr{A}$ at $z_1 = z_2 = 1$.

The problem posed by these indeterminate points of R in $\mathscr{A}$ is not insurmountable and in fact may be avoided by restricting R to the domain $\mathscr{D}' = \mathscr{D} \setminus \mathscr{A}$. In other words R is defined on that part of $\mathscr{D}$ which does not overlap with $\mathscr{A}$.

The second problem is connected with the so-called Cousin's problems (Gunning and Rossi 1965) in several complex variables. The idea of a meromorphic function R on some domain $\mathscr{D}'$ is replaced by an equivalence relation (Lelong 1960) on overlapping neighbourhoods in $\mathscr{D}'$. To see this we remind ourselves of the definition of a rational (meromorphic) function R on $\mathscr{D}'$. We want R on $\mathscr{D}'$ rather than $\mathscr{D}$ to avoid the ambiguities of indeterminate points of R in $\mathscr{D}$. R is a rational (meromorphic) function at each point ζ_0 in $\mathscr{D}'$ if each ζ_0 has a neighbourhood U_{ζ_0} in which P and Q are holomorphic and $R = P/Q$ has a determined value (finite or infinite) in $\mathscr{D}' \cap U_{\zeta_0}$. Thus we can find an open covering of $\mathscr{D}'$ $\{V_i\}$ $i \in I$ (a denumerable index set) on which $R = P_i/Q_i$ and $\bigcup_{i \in I} V_i = \mathscr{D}'$. On overlapping neighbourhoods V_i, V_j with $V_i \cap V_j \neq \phi$ we have $P_i Q_j = P_j Q_i$ and this defines the equivalence relation on $\mathscr{D}'$ at each point.

This problem, however, reduces to a theoretical nicety which for all practical and computational purposes may be ignored.

The third problem is connected with possible use of rational approximants R in continuing an analytically holomorphic function on to a cut or across a cut to other sheets. Algebraic branch points in several complex variables have neighbourhoods which are locally non-euclidean (Gunning and Rossi 1965, Bremermann 1965). Thus to ensure a monodromic extension of the domain of the holomorphic function f by means of its rational approximants R on to a cut, we have to keep away from the branch points.

Of the three problems discussed the final one is the most important in practical terms. For in analogy with the Padé approximants in one dimension, which may be employed (Basdevant 1972) in continuing holomorphic functions analytically outside their domains of convergence and on to cuts, one has to ensure stability of the continuation.

In a neighbourhood of a branch point in two or more variables this may not be guaranteed and therefore such neighbourhoods must be well avoided.

In this paper we have presented a definition of a special type of rational approximant in two variables and a simple convergence theorem based on it. Some of the properties of these approximants will be presented shortly in a separate paper.

Acknowledgments

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Appendix 1

The following is an example indicating the number of equations involved in solving the system. By taking $b_{00} = 1$, the system can be solved. For example $N = 2$, $M = 1$, $N' = 1$, $M' = 0$

$$\left.\begin{aligned} a_{00} &= b_{00}c_{00} \\ a_{10} &= b_{00}c_{10}+b_{10}c_{00} \\ a_{01} &= b_{00}c_{01} \\ a_{11} &= b_{00}c_{11}+b_{10}c_{01} \\ a_{20} &= b_{00}c_{20}+b_{10}c_{10} \\ a_{21} &= b_{00}c_{21}+b_{10}c_{20} \end{aligned}\right\} (N+1)(N'+1) = 3\times 2 = 6$$

$$0 = b_{00}c_{30}+b_{10}c_{20} \qquad M = 1$$

Appendix 2

We present here some solutions of rational approximants for the function

$$f(z_1, z_2) = \frac{z_1\,e^{z_1} - z_2\,e^{z_2}}{z_1 - z_2}$$

which is entire and has the following Taylor expansion in the neighbourhood U of the origin:

$$f(z_1, z_2) = \sum_{\mu,\nu=0}^{\infty} \frac{1}{(\mu+\nu)!} z_1^{\mu} z_2^{\nu}$$

A.2.1. A geometrical representation of the infinite 'cubic' configuration

One way to give a satisfactory geometrical representation of the infinite 'cubic' configuration is by introducing counting procedure for the B-planes. This is to enable us to mark the position on an infinite string, of the point of suspension of each B-plane. This linearizes the B.

Table 1. Solution for $f(z_1, z_2) = (z_1 e^{z_1} - z_2 e^{z_2})(z_1 - z_2)^{-1}$.

z_1		0·25	0·45	0·65	0·85
z_2		0·05	0·25	0·45	0·65
$f(z_1, z_2)$	$\dfrac{z_1 e^{z_1} - z_2 e^{z_2}}{z_1 - z_2}$	1·342213	1·923672	2·696801	3·717990
$[1, 1; 0, 0]_f(\zeta)$	$1 + z_1 + z_2 + \frac{1}{2}z_1 z_2$	1·306249	1·756249	2·246249	2·776249
$[1, 1; 1, 0]_f(\zeta)$	$\dfrac{1 + \frac{1}{2}z_1 + z_2}{1 - \frac{1}{2}z_1}$	1·342855	1·903225	2·629628	3·608693
$[1, 1; 1, 1]_f(\zeta)$	$\dfrac{1 + \frac{1}{2}z_1 + \frac{1}{2}z_2 - \frac{1}{4}z_1 z_2}{1 - \frac{1}{2}z_1 - \frac{1}{2}z_2 + \frac{1}{4}z_1 z_2}$	1·328465	1·752211	2·109012	2·275531
$[2, 2; 1, 1]_f(\zeta)$	$\dfrac{1 + \frac{2}{3}z_1 + \frac{2}{3}z_2 + \frac{39}{240}z_1 z_2 + \frac{1}{6}z_1^2 + \frac{1}{6}z_2^2 - \frac{39}{240}z_1 z_2^2 - \frac{39}{240}z_1^2 z_2 - \frac{97}{1440}z_1^2 z_2^2}{1 - \frac{1}{3}z_1 - \frac{1}{3}z_2 + \frac{1}{240}z_1 z_2}$	1·342342	1·927818	2·729352	3·873792
Chisholm $[1, 1; 1, 1]_f(\zeta)$	$\dfrac{1 + \frac{1}{2}z_1 + \frac{1}{2}z_2 - \frac{1}{6}z_1 z_2}{1 - \frac{1}{2}z_1 - \frac{1}{2}z_2 + \frac{1}{3}z_1 z_2}$	1·343901	1·936362	2·742005	3·818614

$$
\begin{tikzcd}
B_{00} \arrow[r] & B_{01} \arrow[d] & B_{02} \arrow[r] & B_{03} \arrow[d] & \cdots \\
B_{10} \arrow[d] & B_{11} \arrow[l] & B_{12} \arrow[u] & B_{13} \arrow[d] & \cdots \\
B_{20} \arrow[r] & B_{21} \arrow[r] & B_{22} \arrow[u] & B_{23} \arrow[d] & \cdots \\
B_{30} \arrow[d] & B_{31} \arrow[l] & B_{32} \arrow[l] & B_{33} \arrow[l] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

The arrows in the above diagram indicate the order in which the B-planes are positioned on the infinite string. That is: $B_{00}B_{01}B_{11}B_{10}B_{20}B_{21}B_{22}B_{12}B_{02}B_{03}\ldots$

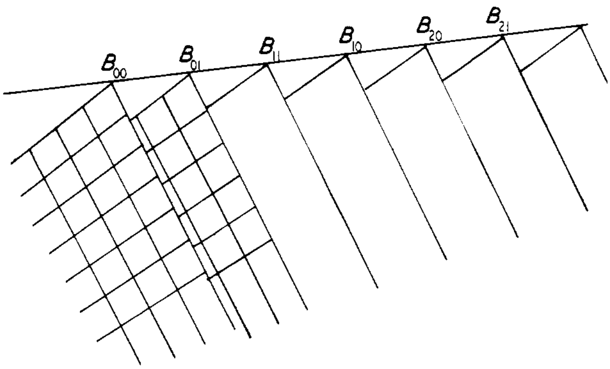

Figure 2.

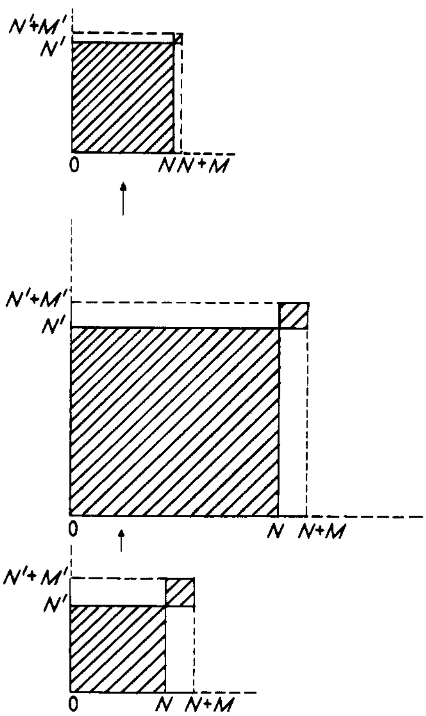

Figure 3. Diagrams showing behaviour as $N \to \infty$, $N' \to \infty$, $N > M$, $N' > M'$.

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